Recursive encoding and decoding of the noiseless subsystem for qudits

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We give a full explanation of the noiseless subsystem that protects a single qubit against collective errors and the corresponding recursive scheme described by C.-K. Li *et al.* [Phys. Rev. A **84**, 044301 (2011)] from a representation theory point of view. Furthermore, we extend the construction to qudits under the influence of collective SU(d) errors. We find that under this recursive scheme, the asymptotic encoding rate is $1/d$.

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I. INTRODUCTION

Quantum computing and quantum information processing make use of quantum systems as computational resources to outperform their classical counterparts. It is expected that a quantum computer solves computationally hard tasks for a classical computer, such as prime-number factorization of a large number, in a practical time and quantum key distribution realizes a 100% secure classical information transmission. In spite of this expectation, a working quantum computer has not become a reality yet. One of the obstacles against its realization is decoherence. Decoherence is a process caused by a coupling between a quantum system (a quantum computer in the present context) and its environment. A pure state to be used as a computational resource becomes a dirty mixed state due to decoherence and then the computational outcome is not reliable any more.

There are several strategies to fight against decoherence, and quantum error correcting codes (QECCs) are one of the best weapons. A pure state may be contaminated due to the interaction between the system and the environment. Then one may embed the quantum information to higher dimensional Hilbert space so that either (i) the error acting on the physical qubit may be identified by introducing error syndrome measurement qubits so that the initial quantum information is recovered after applying appropriate corrections or (ii) the error operator acts only on a part of the Hilbert space, keeping the initial quantum information intact. The second QECC scheme is often called "error-avoiding" coding for this reason. The decoherence-free subspace (DFS) and noiseless subsystem (NS) are two popular examples of the second kind [1–14].

In this paper, we consider the second approach to dealing with quantum channels, in which all physical qubits involved in coding suffer from the same error operators. There are two relevant cases in which such error operators are in action: (i) when the size of a quantum computer is much smaller than the wavelength of the external disturbances and (ii) when photonic qubits are sent one by one through an optical fiber with a fixed imperfection. In both cases, the qubits suffer from the same errors, leading to decoherence. Another instance in which such encoding is useful is when Alice sends quantum information to Bob (possibly billions of light years away) without knowing which basis vectors Bob employs. Then mismatching of the basis vectors is common for all qubits and such mismatching is regarded as collective noise.

In our previous publications, we reported the following results:

(1) For a limited class of error operators $\{\sigma_x^{\otimes n},\sigma_y^{\otimes n},\sigma_z^{\otimes n}\}$, it is possible to iteratively implement encoding/decoding circuits, which protects $n-1$ logical qubits when n is odd and $n-2$ logical qubits when n is even [15]. When n physical qubits protect k logical qubits, the encoding rate is defined by k/n. The asymptotic encoding rate obtained in [15] is 1, as $n \gg 1$ for both cases.

(2) For general error operators $W^{\otimes n}$, where $W \in \mathrm{SU}(2)$, we gave explicit recursive implementation of encoding/decoding circuits for arbitrary numbers n of physical qubits. We have shown that $n = 2k+1$ physical qubits protect k logical qubits, leading to the asymptotic encoding rate of $1/2$ [16].

(3) A qudit is a d-dimensional analog of a qubit. It transforms under the action of the fundamental representation of SU(d). [It should not be confused with a vector transforming under the action of a d-dimensional representation of SU(2).] In [17], we identified the subspace with the maximal dimension of the total Hilbert space of physical qudits when $d=2$ and 3, which is immune to collective noise operators of the form $W^{\otimes n}$, where $W \in$ SU(d) ($d = 2,3$). It was shown that the encoding rate approaches 1 as $n \gg 1$. The irreducible representation (irrep) giving the encoding subspace with the maximal dimension is given by an almost-rectangular Young tableau [17]. Identification of an irrep with the maximal multiplicity for $d > 3$ is a highly nontrivial open problem even though the decomposition of $W^{\otimes n}$ into irreps is well established.

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In the present paper, we demonstrate why the recursion relation introduced in [16] works from the representation theory point of view and generalize this relation to the qudit case. We show how to implement encoding/decoding circuits for n physical qudits, which results in the asymptotic encoding rate of $1/d$. A natural question arising from this statement must be 'Why do we do this analysis even though it is known that there is a DFS-NS which gives an asymptotic encoding rate of 1?' To implement encoding/decoding circuits, we need quantum circuits, which *physically* represent the encoding/decoding matrix $U_E/U_E^\dagger$. Although it may be possible to find the quantum circuits for small n by some trial and error, it is totally impossible to find them if the number of qudits n is more than 100 or even 10. We believe that recursive implementation of the circuits is the only possible way to physically realize the proposed scheme.

The rest of the paper is organized as follows. In the next section, we outline the results of [16] for qubits from a representation theoretical viewpoint so that they can be easily generalized to the qudit case. In Sec. III, we give a detailed analysis of recursive implementation of qudit encoding/decoding circuits and prove that this implementation gives the asymptotic encoding rate of $1/d$. Section IV is devoted to a summary and discussion.

II. SU(2) RECURSION RELATION REVISITED

In this section, we review and give further explanation of the three-qubit noiseless subsystem and recursion relation described in [16] from a representation theory point of view. This approach has the advantage of being general and applicable to systems with d levels.

Let us denote the error acting on a single site $W \in \mathrm{SU}(2)$ and the total collective noise on the system $\mathcal{E} = W \otimes W \otimes W$. Such an operation is totally symmetric under exchanges, and the resulting 8×8 matrix is reducible as $\mathbf{4} + \mathbf{2} + \mathbf{2}$. In the context of representation theory, irreps of groups are conveniently labeled by the Young tableau. The fundamental irrep of SU(2) is labeled $\begin{array}{|c|}\hline 1\\\hline\end{array}$. The form of the reduction is contained in the expansion [18]

$$\begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline\end{array} = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \oplus \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}. \tag{1}$$

The irreps on the right-hand side have the dimensions 4, 2, and 2, respectively. The two copies of the fundamental irrep give rise to a noiseless subsystem. These irreps are more commonly known as spin-3/2 and spin-1/2 representations of SU(2), respectively. The dimension of an irrep is the number of vectors belonging to it whose entries are the Clebsch-Gordan coefficients, and they are sometimes called Young-Yamanouchi vectors [19].

If we denote the elements of the fundamental irrep u and d (or $|u\rangle$, $|d\rangle$), which refer to the spin-up and spin-down states, the vectors belonging to the irreps that appear in Eq. (1) can be written as

$$\begin{array}{l}
\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\left\{\begin{array}{l}\frac{1}{\sqrt{6}}(-[ud+du]u+2[uu]d)\\ \frac{1}{\sqrt{6}}([ud+du]d-2[dd]u)\end{array}\right.\\
\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\left\{\begin{array}{l}\frac{1}{\sqrt{2}}(ud-du)u\\ \frac{1}{\sqrt{2}}(ud-du)d\end{array}\right.\\
\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\left\{\begin{array}{l}uuu\\ \frac{1}{\sqrt{3}}(uud+udu+duu)\\ \frac{1}{\sqrt{3}}(ddu+dud+udd)\\ ddd\end{array}\right.
\end{array} \tag{2}$$

The unitary transformation U_E that block-diagonalizes $\mathcal{E}$ as $W \oplus W \oplus W^{(3/2)}$, where $W^{(3/2)}$ is the spin-3/2 representation of W, is constructed by using these basis vectors as columns and grouping them in a proper fashion such as in [20]. Here, the vertical ellipses indicate that vectors of the irrep are placed as column vectors:

$$U_E = \begin{pmatrix} \vdots & \vdots & \vdots \\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} & \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \\ \vdots & \vdots & \vdots \end{pmatrix}. \tag{3}$$

An element of SU(2) is naturally expressed in exponential form as $e^{i(r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)}$. Different representations can be obtained by replacing Pauli matrices, which correspond to the fundamental representation, with larger representations of the algebra $\mathfrak{su}(2)$. In the particular case of the four-dimensional irrep, the generators are given as (see, for example, [21,22])

$$\begin{aligned}
J_x^{(3/2)} &= \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix},\\
J_y^{(3/2)} &= i\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix},\\
J_z^{(3/2)} &= \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.
\end{aligned} \tag{4}$$

Figure 1 shows the entire operation of sending state $|u\rangle\,|\psi\rangle\,|v\rangle$ through the noisy channel. Here, $|u\rangle$ is the spin-up state, $|\psi\rangle$ represents the data, and $|v\rangle$ is an arbitrary ancillary. This particular choice of the input state as well as the output given in this figure can be justified in the following manner. The density matrix of the system can be written

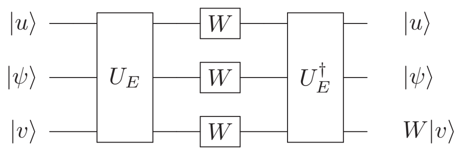

FIG. 1. Reordered version of the three-qubit QECC from [16]. A circuit representation for U_E is shown in Fig. 2. $|\psi\rangle$ represents the data qubit. $|v\rangle$ is the state of the ancillary qubit, which can be arbitrary.

as $(|\psi\rangle\langle\psi| \otimes |v\rangle\langle v|) \oplus 0_4$. The action of the reduced error operator $\mathcal{E}' = U_E^\dagger \mathcal{E} U_E = (\mathbb{1}_2 \otimes W) \oplus W^{(3/2)}$ on this state only rotates $|v\rangle$, leaving $|\psi\rangle$ intact. Here 0_m is an $m \times m$ zero matrix and $\otimes_m$ is an $m \times m$ identity matrix. An important corollary is that the action of $\mathcal{E}'$ on the subspace $|u\rangle|\psi\rangle|v\rangle$ is equivalent to $\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes W$. This enables recursive construction of a noiseless subsystem for $2k+1$ qubits.

For instance, to construct a noiseless subsystem for a five-qubit system, we use U_E twice as shown in Fig. 3. By replacing the dashed part ($\mathcal{E}'$) of the circuit with $\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes W$, we obtain the circuit shown in Fig. 4. If we repeat the process for the lower three-qubits, it becomes clear that the output is $|u\rangle|\psi_2\rangle|u\rangle|\psi_1\rangle(W|v\rangle)$ and the states $|\psi_i\rangle$, $i = 1,2$, are protected against noise (Fig. 5).

III. RECURSIVE CONSTRUCTION OF A NOISELESS SUBSYSTEM FOR QUDITS

Now that we have the tools for a general analysis, we turn to the problem of finding an analogous recursion scheme for d-level systems. To process, we first need to determine the number of qudits m we require to avoid collective noise $\mathcal{E} = W^{\otimes m}$, where W is an arbitrary error operator on a single qudit and an element of the fundamental representation of SU(d). To construct a noiseless subsystem, we must have an irrep with a multiplicity of at least d. It turns out that the fundamental irrep appears exactly d times for $d+1$ qudits, which can be shown by using the Frobenius formula [19]:

$$(d+1)!\frac{\prod_{1\leqslant i<j\leqslant d} \nu_i - \nu_j + j - i}{\prod_{i=1}^{d}(\nu_i + d - i)!} = d. \qquad (5)$$

Here, ν_i denote the row lengths of the corresponding Young diagram in top-to-bottom order. Such a noiseless subsystem can protect a single logical qudit against errors.

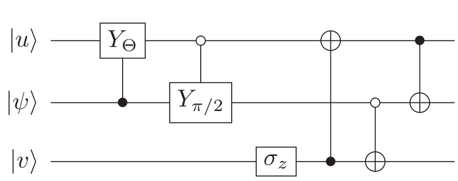

FIG. 2. Reordered SU(2) encoding gate U_E from [16], in terms of single-qubit and two-qubit controlled-U gates. $Y_\theta = \exp(i\sigma_y\theta)$ and $\sin\Theta = \sqrt{2/3}$.

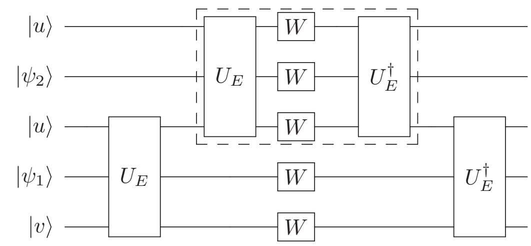

FIG. 3. Recursive five-qubit circuit diagram from [16]. In this reordered version, the gates act on neighboring three-qubits only.

The collective error operator $\mathcal{E}$ is block-diagonalized by a unitary transformation $U_E^\dagger$ as

$$\mathcal{E}' = (\mathbb{1}_d \otimes W) \oplus \mathcal{O}, \qquad (6)$$

where $\mathcal{O}$ represents the direct sum of the remaining representations of W, which are not relevant for our purposes. The transformation matrix, which is the encoding circuit at the same time, is constructed by placing the Young-Yamanouchi vectors [23] of the corresponding irreps as columns below:

$$U_E = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \boxed{1} & \boxed{1} & \cdots & \boxed{1} & \text{other irreps} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \qquad (7)$$

Here each $\boxed{1}$ denotes any irrep that is equivalent to the d-dimensional fundamental irrep, and their ordering is not important and can be treated as a freedom during the construction of the encoding/decoding circuits. In total, there are d copies of this irrep. The vectors belonging to other irreps can be placed in an arbitrary manner. Note that, in practice, we do not need to worry about these vectors as long as they are orthogonal to the basis vectors belonging to the fundamental irreps. Such an orthogonalization is enforced by the unitarity of the encoding circuit.

The proper input state turns out to be

$$|\Psi\rangle = |u\rangle^{\otimes d-1}|\psi\rangle|v\rangle, \qquad (8)$$

where $|\psi\rangle$ is the qudit state carrying the information, $|v\rangle$ is an ancillary qudit prepared in an arbitrary state, and $|u\rangle$ is the d-dimensional vector $(1,0,\ldots,0)^T$, the highest-weight state in the fundamental representation of SU(d). Note that the encoding and decoding can be seen as a basis transformation.

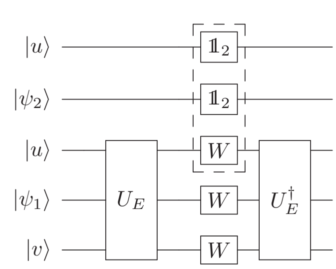

FIG. 4. Schematic recursion. This reduced circuit is equivalent to the one shown in Fig. 3 due to the equivalence given in the text.

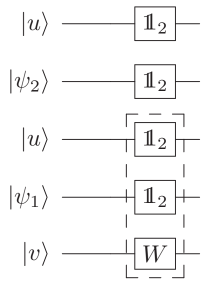

FIG. 5. Final version of the circuit given in Fig. 3.

In this view, $|\Psi\rangle$ has nonzero entries only in the first d^2 rows, which means that such a state belongs to the direct-sum space of the fundamental irreps.

The action of the collective error $\mathcal{E}'$ on this state can be seen by acting on the corresponding density matrix $\rho = (|\psi\rangle\langle\psi| \otimes |v\rangle\langle v|) \oplus 0_q$, where $q = d^{d+1} - d^2$. Clearly, $|v\rangle$ will be distorted into $W|v\rangle$ during the transmission through the noisy channel while the remaining qubits are left intact. We observe that the action of $\mathcal{E}'$ on this subspace is equivalent to $\mathbb{1}_d^{\otimes d} \otimes W$; that is,

$$(U_E^\dagger W^{\otimes d+1} U_E)\,|u\rangle^{\otimes d-1}\,|\psi\rangle\,|v\rangle = |u\rangle^{\otimes d-1}\,|\psi\rangle\, W\,|v\rangle \quad (9)$$

holds. Following the arguments regarding Figs. 3–5, we see that the equivalence enables recursive construction of a $kd+1$-qudit QECC, which is capable of protecting k qudits.

A naive way of constructing noiseless subsystem for k qudits would be to vertically clone the elementary circuit such as the one given in Fig. 1. Since the elementary circuit protects a single qudit using $d+1$ qudits, the asymptotic encoding rate would be $1/(d+1)$. However, with the recursive scheme, given that the number of correctable qudits using $n = kd+1$ for the channel is k, we find the asymptotic behavior of the encoding rate to be $k/n \to 1/d$ as $n \to \infty$ for a fixed d.

IV. CONCLUSION

The noiseless subsystem is a method of using the inherent permutation symmetry of the noise to protect a subsystem against errors. In this work, we have used several powerful tools from representation theory for a better understanding and further generalization of the recursive construction of a subsystem for qubits and have extended our results to qudits. Our approach is based on a $d+1$–qudit encoding circuit whose implementation is realized by the vectors in the fundamental irrep $\boxed{1}$. It should be noted that different constructions based on different irreps are possible [14], although they may not necessarily be suitable for our recursive scheme. We have then generalized our construction to $n = kd+1$. Encoding/decoding can be realized using $U_E/U_E^\dagger$ successively, operating on $d+1$ neighboring qudits at a time, which can be of practical importance.

We note, however, that our construction does not give the maximum number of correctable qudits for the channel. When the irrep with maximal degeneracy is used instead of the fundamental representation, the ratio of protected qudits to total number of qudits is $k/n \to 1$ as $n \to \infty$ [17]. However, even though the DFS-NS with the maximal dimension is identified, we do not yet know how to implement the encoding circuit efficiently. Our study here gives a foolproof implementation of the encoding circuit, although the efficiency is $1/d$ for qudits. It is certainly desirable to find a recursion relation for maximal dimension DFS-NS, which is left for future work.

It should be emphasized that the decomposition for U_E given in Fig. 2 is not canonical. In general, given a universal set of elementary gates, the U_E matrix can be decomposed in infinitely many different ways. Each decomposition has its trade-offs; some will require less energy or operational time than others, for instance. Identification of "good" elementary gates (which are not necessarily one- or two-qudit gates [24]) and optimization of the decomposition in terms of these gates with respect to a cost function both require us to specify a Hamiltonian. Hence, both problems are implementation dependent and no optimal generic decomposition exists. Once the Hamiltonian is decided upon, obtaining an optimal decomposition is still a nontrivial problem [25].

Finally, we remark that our scheme is applicable to nonunitary error channels as well. The essential ingredient for our construction is the permutation symmetry of the collective error operator $\mathcal{E}$, and the Kraus operator W may belong to a Lie group G other than SU(d) whose fundamental representation is d-dimensional, such as SL($d,\mathbb{C}$), following the Schur-Weyl duality. That is, the U_E given in Eq. (7) will block-diagonalize $\mathcal{E}$ when $W \in$ SL($d,\mathbb{C}$) and the resulting block structure will be the same [26].

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